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General Ginsparg-Wilson fermions and index

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Abstract

We show rigorously that for general Ginsparg-Wilson fermions the dimensions of the geometric eigenspace and of the algebraic one for zero modes agree so that the index theorem on the lattice is not spoiled by unwanted additional terms.

In recent years the formulation of the index theorem on lattices has become possible [1, 2, 3]. It has been shown [4] that the Dirac operator D in [1] obeys the form $\{\gamma_5, D\} = 2D\gamma_5 D$ of the Ginsparg-Wilson (GW) relation [5]. This form has also been assumed in [3]. On the other hand, in [2] the index theorem has been claimed to hold for D satisfying the more general GW relation [5]

$$\{\gamma_5, D\} = 2D\gamma_5 RD \tag{1}$$

where R is a hermitean operator which commutes with γ-matrices. The derivation given there relies on the eigenvectors, i.e. is solely based on the consideration of the geometric eigenspace. However, in general the dimension of the geometric eigenspace can be smaller than that of the algebraic eigenspace, in which case unwanted additional terms would spoil the index theorem. In view of the many works relying on the general form (1), it appears important to clarify whether in that case such a difference occurs. In the present letter we find that settling this question requires a detailed study of properties of the eigennilpotents and show rigorously that the respective dimensions for zero modes agree.

The index theorem follows from the global chiral Ward identity. This identity is obtained requiring $\frac{\mathrm{d}}{\mathrm{d}\eta}\int[\mathrm{d}\bar\psi'\mathrm{d}\psi']\mathrm{e}^{-S_\mathrm{f}'}\mathcal{O}'|_{\eta=0} = 0$ for the transformation $\psi' = \exp(i\eta\gamma_5)\psi$, $\bar\psi' = \bar\psi\exp(i\eta\gamma_5)$. Since we have to deal with the case where the Dirac operator D has zero modes, we must make sure to account properly for them. We therefore replace D by $D-\zeta$ with the parameter ζ being in the resolvent set (i.e. not in the spectrum of D) allowing ζ to go to zero only in the final result. With $S_\mathrm{f}' = \bar\psi'(D-\zeta)\psi'$ we thus obtain

$$\frac{1}{2}\mathrm{Tr}\Big((D-\zeta)^{-1}\{\gamma_5, D\}\Big) - \zeta\,\mathrm{Tr}\Big((D-\zeta)^{-1}\gamma_5\Big) = 0\;. \tag{2}$$

Of course, the validity of the identity (2) can also be verified directly. Obviously it is just a particular decomposition of $\mathrm{Tr}\,\gamma_5 = 0$.

In order to evaluate (2) we use the fact that the resolvent of D is given by [6]

$$(D-\zeta)^{-1} = -\sum_{j=1}^{s}\Big((\zeta-\lambda_j)^{-1}P_j + \sum_{k=1}^{d_j-1}(\zeta-\lambda_j)^{-k-1}Q_j^k\Big)\;. \tag{3}$$

The operators P_j and Q_j in (3) satisfy $P_jP_l = \delta_{jl}P_j$, $P_jQ_l = Q_lP_j = \delta_{jl}Q_j$, and $Q_jQ_l = 0$ for $j \neq l$. The P_j project on the algebraic eigenspaces M_j with dimensions $d_j = \mathrm{Tr}\,P_j$. The Q_j have the property $Q_j^{d_j} = 0$, i.e. they are nilpotents. In terms of these operators the spectral representation of D becomes [6]

$$D = \sum_j(\lambda_jP_j + Q_j)\;. \tag{4}$$

The direct sum of the spaces M_j makes up the total space. We emphasize that they in general are not orthogonal. Correspondingly then in general P_j, which projects on M_j along $\tilde M_j = M_1\oplus\ldots\oplus M_{j-1}\oplus M_{j+1}\oplus\ldots\oplus M_s$, is also not orthogonal and one has $P_j^\dagger \neq P_j$ since $\tilde M_j$ is not the orthogonal complement of M_j. Therefore in the following we have to work carefully with general projections and subspaces.

By inserting (1) into the identity (2) we obtain

$$\mathrm{Tr}(\gamma_5RD) - \zeta\,\mathrm{Tr}\Big(\gamma_5(D-\zeta)^{-1}\Big) + \zeta^2\,\mathrm{Tr}\Big(\gamma_5R(D-\zeta)^{-1}\Big) = 0\;. \tag{5}$$

Dividing (5) by ζ, expressing $(D-\zeta)^{-1}$ by (3), and integrating over ζ around a circle enclosing only the eigenvalue $\lambda_k = 0$ we find

$$\mathrm{Tr}(\gamma_5RD) + \mathrm{Tr}\Big(\gamma_5(P_k + RQ_k)\Big) = 0 \quad \text{for} \quad \lambda_k = 0\;. \tag{6}$$

To evaluate (6) further we need more information about the nilpotents Q_j. They account for the fact that for $d_j > 1$ the dimension g_j of the geometric eigenspace can be smaller than the dimension d_j of the algebraic one. The dimension d_j equals the multiplicity of the solution $\lambda = \lambda_j$ of $\det(D - \lambda 1\!\!1) = 0$ while g_j is given by $g_j =$ dim ker $(D - \lambda_j 1\!\!1)$. From the latter relation and (4) we obtain

$$\text{rank } Q_j = d_j - g_j \ . \tag{7}$$

Furthermore, since $\det(D - \lambda_j 1\!\!1) = 0$ one has $g_j =$ dim ker $(D - \lambda_j 1\!\!1) \geq 1$ and therefore rank $Q_j \leq d_j - 1$ and $1 \leq g_j \leq d_j$.

To specify the spaces in more detail we note that the linearly independent set of eigenvectors f_{jl} with $Df_{jl} = \lambda_j f_{jl}$ and $l = 1, \dots, g_j$ spans the geometric eigenspace M'_j of D with dimension g_j while P_j as introduced above projects on the algebraic one M_j with dimension d_j. For $g_j < d_j$ we choose M''_j such that $M_j = M'_j \bigoplus M''_j$. This defines projections P'_j and P''_j where P'_j projects on M'_j along $M''_j \bigoplus \tilde{M}_j$ and P''_j on M''_j along $M'_j \bigoplus \tilde{M}_j$. One then has

$$P_j = P'_j + P''_j \tag{8}$$

with $P'_j P''_j = P''_j P'_j = 0$ and Tr $P'_j = g_j$.

It will be important that with this decomposition according to (4) we have

$$Q_j \phi = 0 \quad \text{for} \quad \phi \in M'_j \tag{9}$$

characteristic for the geometric eigenspace. In terms of operators this means that

$$Q_j P'_j = 0 \quad \text{and} \quad Q_k P''_j = Q_j \ . \tag{10}$$

On the other hand, for the products $P'_j Q_j$ and $P''_j Q_j$ in general various results can occur which only have to satisfy

$$P'_j Q_j + P''_j Q_j = Q_j \tag{11}$$

as is needed because of $P_j Q_j = Q_j$.

We next impose the usual requirement of γ_5-hermiticity of D,

$$D^\dagger = \gamma_5 D \gamma_5 \ . \tag{12}$$

Then the GW relation (1) can be written as $D + D^\dagger = 2D^\dagger R D$ which implies that

$$Df_{kl} = 0 \quad \text{and} \quad D^\dagger f_{kl} = 0 \quad \text{for} \quad \lambda_k = 0 \tag{13}$$

hold simultaneously.

From (13) and (12) one obtains $[\gamma_5, D] f_{kl} = 0$ for $\lambda_k = 0$. Therefore the f_{kl} can be chosen such that $\gamma_5 f_{kl} = s_{kl} f_{kl}$ with $s_{kl} = \pm 1$. The sets of f_{kl} with $s_{kl} = +1$ and $s_{kl} = -1$ then define subspaces $M_k^{\prime(+)}$ and $M_k^{\prime(-)}$, respectively, with $M'_k = M_k^{\prime(+)} \bigoplus M_k^{\prime(-)}$. This in turn defines the projections $P_k^{\prime(+)}$ on $M_k^{\prime(+)}$ along $M_k^{\prime(-)} \bigoplus M''_k \bigoplus \tilde{M}_k$ and $P_k^{\prime(-)}$ on $M_k^{\prime(-)}$ along $M_k^{\prime(+)} \bigoplus M''_k \bigoplus \tilde{M}$. We thus get

$$P'_k = P_k^{\prime(+)} + P_k^{\prime(-)} \tag{14}$$

with $P_k'^{(+)}P_k'^{(-)} = P_k'^{(-)}P_k'^{(+)} = 0$ and Tr $P_k'^{(\pm)} = g_k^{(\pm)}$, where $g_k^{(\pm)}$ denotes the numbers of modes with $s_{kl} = \pm 1$, respectively, and $g_k = g_k^{(+)} + g_k^{(-)}$.

By inserting (8) and (14) into (6) we now obtain the more detailed form

$$\mathrm{Tr}(\gamma_5 RD) + g_k^{(+)} - g_k^{(-)} + \mathrm{Tr}(\gamma_5 P_k'') + \mathrm{Tr}(\gamma_5 RQ_k) = 0 \quad \text{for} \quad \lambda_k = 0\,. \tag{15}$$

Comparing (15) with the relation given in [2] we see that the terms

$$+\mathrm{Tr}(\gamma_5 P_k'') + \mathrm{Tr}(\gamma_5 RQ_k) \tag{16}$$

are missing there. The occurrence of the terms (16) is related to the possibility that the dimension of the geometric eigenspace can be smaller than that of the algebraic one. In fact, according to (7) we have rank $Q_k = d_k - g_k$ and from (8) it follows that Tr $P_k'' = d_k - g_k$. Thus it becomes obvious that it is $g_k = d_k$ which is needed to make (16) vanish. This means that, in order to get rid of the unwanted terms (16), D should have a property which guarantees that the solutions of the eigenvalue problem satisfy $g_k = d_k$.

It can be shown [6] that if D is normal, $[D, D^\dagger] = 0$, one has $Q_j = 0$ for all j. Thus, for such D one gets $g_j = d_j$ for all j and, in particular, $g_k = d_k$ for $\lambda_k = 0$. In addition with normality of D one has [6] $P_j^\dagger = P_j$ for all j, i.e. orthogonality of the eigenprojections and of the associated subspaces. However, a Dirac operator satisfying (1) is in general not normal. To make this explicit we note that from (1) using (12) and $[\gamma_5, R] = 0$ one obtains $[D, D^\dagger] = 2D^\dagger[R, D]D^\dagger$. Thus it is seen that one would need $[R, D] = 0$ to make D normal, which to fulfil generally would require R to be a multiple of the identity, i.e. to restrict to the simple form $\{\gamma_5, D\} = 2D\gamma_5 D$ of the GW relation.

To proceed with the more general relation (1) we note that $Q_k = 0$ is actually needed for $\lambda_k = 0$ only. In the following we prove this weaker property. For this purpose we first observe that from (13) with (9) we have

$$Q_k P_k' = 0 \quad \text{and} \quad Q_k^\dagger P_k' = 0 \quad \text{for} \quad \lambda_k = 0\,. \tag{17}$$

We next remember that P_k' projects on M_k' along $M_k'' \oplus \tilde{M}_k$ so that by definition $(P_k')^\dagger$ projects on $(M_k'' \oplus \tilde{M}_k)^\perp$ along $M_k'^\perp$ where $\perp$ denotes orthogonal complements. On the other hand, from (13) it follows that P_k' and $(P_k')^\dagger$ both project on M_k'. Therefore we get $M_k' = (M_k'' \oplus \tilde{M}_k)^\perp$ which implies

$$(P_k')^\dagger = P_k' \quad \text{for} \quad \lambda_k = 0\,. \tag{18}$$

From this and the adjoint of the second relation in (17) we now obtain $P_k' Q_k = 0$ and thus, with (11), arrive at the relations

$$P_k' Q_k = 0 \quad \text{and} \quad P_k'' Q_k = Q_k \quad \text{for} \quad \lambda_k = 0\,. \tag{19}$$

Obviously (19) can be satisfied with $Q_k = 0$. This is, however, not possible with $Q_k \neq 0$, as we shall show below, which will complete our proof.

In case of $Q_j \neq 0$, for some integer n with $1 \leq n \leq d_j - 1$ we have $Q_j^{n+1} = 0$ but $Q_j^n \neq 0$. Then, since the range of Q_j^n is nonzero, there is a nonzero vector ϕ which satisfies $Q_j \phi = 0$. Further, there is some vector ψ_1 such that $\phi = Q_j^n \psi_1$. For $n > 1$ we can successively define

$$\psi_\nu = Q_j \psi_{\nu-1} \quad \text{for} \quad \nu = 2, \ldots, n \tag{20}$$

and we get $\phi = Q_j \psi_n$ for $n \geq 1$. Decomposing ψ_ν according to

$$\psi_\nu = \varphi_\nu + \chi_\nu \quad \text{with} \quad \varphi_\nu \in M'_j\ ,\ \chi_\nu \in M''_j \tag{21}$$

and noting that by (9) $Q_j \varphi_\nu = 0$ these relations become

$$\chi_\nu = Q_j \chi_{\nu-1} \quad \text{with} \quad \nu = 2, \ldots, n \quad \text{for} \quad n > 1 \tag{22}$$

and $\phi = Q_j \chi_n$ for $n \geq 1$. Because of $Q_j \phi = 0$ by (9) we have $\phi \in M'_j$. Thus it is seen that the nilpotent property of Q_j relies on a sequence of $n-1$ transformations within M''_j, one from M''_j to M'_j, and a final one which then according to (9) gives zero. Obviously it is crucial for this property that the indicated mapping from M''_j to M'_j is possible.

The χ_μ with $\mu = 1, \ldots, n$ for $n = r_j$ provide a basis of M''_j and for $n < r_j$ a basis of a n-dimensional subspace of M''_j. In the latter case we repeat the above procedure for the remaining $(r_j - n)$–dimensional subspace of M''_j in which $Q_j^{n_2+1} = 0$ but $Q_j^{n_2} \neq 0$ for some integer n_2 with $1 \leq n_2 \leq n_1 \equiv n$. Possibly further repetitions are needed until the space M''_j is exhausted and we reach $n_1 + n_2 + \ldots + n_h = r$ with $n_1 \geq n_2 \geq \ldots \geq n_h \geq 1$ and $1 \leq h \leq g_j$. Clearly in each of the subspaces of M''_j involved in this process it remains crucial that M'_j can be reached by the transformations in the indicated way.

Because the range of P''_k is M''_k, the condition $P''_k Q_k = Q_k$ for $\lambda_k = 0$ of (19) implies that the range of Q_k must be within M''_k. Therefore it cannot map to M'_k as we have shown above to be necessary for the nilpotent property of $Q_k \neq 0$. Thus $Q_k \neq 0$ is excluded and (19) is indeed only satisfied by $Q_k = 0$, which completes the proof.

With $Q_k = 0$ by (7) we now have $g_k = d_k$ and $P''_k = 0$ for $\lambda_k = 0$ and arrive at the result that the terms (16) vanish. Further, because of $P'_k = P_k$, as for M'_k before, the decomposition $M_k = M_k^{(+)} \bigoplus M_k^{(-)}$ with the projections $P_k^{(+)}$ and $P_k^{(-)}$ can be introduced and one sees that $g_k^{(\pm)} = d_k^{(\pm)}$ holds. With $P'_k = P_k$ from (18) in addition $P_k^\dagger = P_k$ for $\lambda_k = 0$ gets obvious.

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